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RESEARCH ARTICLE

Fractals in trade duration: capturing long-range dependence and heavy tailedness in modeling trade duration

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Abstract Several studies that have investigated a few stocks have found that the spacing between consecutive financial transactions (referred to as trade duration) tend to exhibit long-range dependence, heavy tailedness, and clustering. In this study, we empirically investigate whether a larger sample of stocks exhibit those characteristics. We do so by comparing goodness of fit in modeling trade duration data for stable distribution and fractional stable noise based on a procedure applying bootstrap

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methods developed by the authors with several alternative distributional assumptions in modeling trade duration data. The empirical results suggest that the autoregressive conditional duration model with stable distribution fits better than other combinations, while fractional stable noise itself fits better for the time series of trade duration. Our result is consistent with the general findings in the literature that trade duration is informative and that short trade durations move prices more than long trade duration. In addition, our result confirms the advantage of fractal models in the study of roughness in trade duration and provides some evidence for duration dependence.

Keywords Fractal processes · Point processes · Self-Similarity · Stable distribution · Trade duration

JEL Classification C41 · G14

1 Introduction

There is considerable interest in the information content and implications of the time between consecutive financial transactions (referred to as trade duration) for trading strategies and intra-day risk management. Market microstructure theory, supported by empirical evidence, suggests that the spacing between trades be treated as a variable to be explained or predicted since time carries information and closely correlates with price volatility (see Bauwens and Veredas 2004; Diamond and Verrecchia 1987; Engle 2000; Engle and Russell 1998; Hasbrouck 1996; O'Hara 1995). Empirically it has been found that returns and volatility directly interact with trade duration and trade order size; and, short duration moves price more than long duration across stocks and across time (see Manganelli 2005; Furfine 2007). Several studies (for example, Bauwens and Giot 2000; Dufour and Engle 2000; Engle and Russell 1998; Jasiak 1998) report that trade duration tends to exhibit long-range dependence (i.e., trade duration tends to be persistent), heavy tailedness (i.e., extremely short or long trade duration can be often observed), and clustering (i.e., short duration follows short duration and long duration follows long duration).

These findings raise two questions in studying the information content of trade duration that we address in this paper:

1. Can single stochastic processes which capture long-range dependence and heavy tailedness be used in modeling trade duration data ?
2. Can a relatively "powerful" distributional assumption in a relatively "simple" functional structure be used for efficient modeling of trade duration data?

It is necessary to treat long-range dependence, heavy tailedness, and clustering simultaneously in order to obtain more accurate predictions. Rachev and Mittnik (2000) note that for modeling financial data, not only does model structure play an important role, but distributional assumptions influence the modeling accuracy. The stable Paretian distribution[1] can be used to capture characteristics of trade duration since it is rich enough to encompass those stylized facts in such data, such as non-Guassian, heavy tails, long-range dependence, and clustering. Other researchers have shown the advantages of stable distributions in financial modeling (see Fama 1963; Mittnik and Rachev 1993; Rachev and Mittnik 2000; Sun et al. 2007a). Meanwhile several studies have reported that long-range dependence, self-similar processes, and stable distribution are very closely related (see Doukhan et al. 2003; Rachev and Mittnik 2000; Rachev and Samorodnitsky 2001; Samorodnitsky and Taqqu 1994; Sun et al. 2007b).

The Hurst index is also used to model long-range dependence (see Hurst 1951, 1955). It quantifies the degree of long-range dependence and measures the self-similarity scaling. Fortunately, one type of self-similar process can possess the Hurst index and stable distribution together (i.e., can capture both long-range dependence and heavy tailedness). This kind of stochastic process is a fractional stable noise generated from fractional stable motion (see Samorodnitsky and Taqqu 1994). Therefore, single stochastic processes can capture long-range dependence and heavy tailedness, answering the first question posed above. Based on estimating intensity of point processes, an autoregressive conditional duration (ACD) model is proposed by Engle and Russell (1998) for modeling trade duration with intertemporal correlation. The ACD model is a joint approach combining transition analysis and Engle's autoregressive conditional heteroscedasticity (ARCH) model. The motivation behind the ACD and the ARCH models is that in financial markets events tend to occur in clusters. If fractional stable noise can be subordinated into the functional structure of the ACD model, then the second question can be answered.

In order to answer the two questions posed above, this paper introduces fractional stable noise as the single stochastic processes to model trade duration. In the empirical analysis of this paper, fractional stable noise is subordinated to the ACD model to model trade duration. As to self-similar processes, other single stochastic processes, such as fractional Gaussian noise which captures long-range dependence, are also introduced as an alternative. Since the stable distribution itself can capture heavy tailedness and long-range dependence, we propose it as an alternative distribution that can better explain trade duration. In the empirical analysis, stable distribution is also subordinated to the ACD model. Some other distributions that are often used in modeling trade duration, such as lognormal distribution, exponential distribution and Weibull distribution, have been selected as alternative distributional assumptions in order to compare goodness of fit with the stable distribution and fractional stable noise. Utilizing two test statistics usually used to evaluate model performance under heavy-tailed assumptions, we examine trade duration for a sample of stocks to compare which distributional assumption fits better. By applying a newly developed test procedure that we formulate, based on a bootstrap method, we obtain empirical results that suggest the

[1] To distinguish between Gaussian and non-Gaussian stable distribution, the latter is usually named stable Paretian distribution or Lévy stable distribution. Referring to it as a stable Paretian distribution highlights the fact that the tails of the non-Gaussian stable density have Pareto power-type decay and Lévy stable is the recognition of pioneering works done by Paul Lévy to the characterization of non-Gaussian stable laws (see Rachev and Mittnik 2000).

fractional stable noise and stable distribution dominate these alternative assumptions with high statistical significance. Comparing goodness of fit in the modeling of trade duration data for stable distribution and fractional stable noise, the empirical results indicate that the ACD model with stable distribution fits better than other combinations, while fractional stable noise itself fits better for the time series of trade duration.

The contributions of our paper to the microstructure finance literature are threefold. The first, and the major contribution, is that we are the first to employ fractal models in the study of roughness[2] of trade duration. We present new applications of fractal processes to model trade duration movements with a large sample. Our study confirms the advantages of fractal models in analyzing roughness of trade duration compared to other models, since “the proper language of the theory of roughness in nature and culture is fractal geometry” (see Mandelbrot 2005, p.193). Second, we extend the ACD model with fractal models and find results that are consistent with other studies that have shown that trade duration is informative and correlated with market volatility. Fractal processes have been shown to be superior to other models in modeling equity market intra-daily return dynamics (see Sun et al. 2007b). The stock returns and their durations exhibit similar movement captured by fractal models, implying a certain inter-relationship between market volatility and information contained in trade duration. This is partially explained by the use of the ACD-GARCH model in the study of ultra-high-frequency data (see Ghysels and Jasiak 1998). Third, trade durations of different stocks exhibit dependence and such dependence is possibly caused by new information revealed from trade intensity (see Simonsen 2007). In our study, we provide some evidence to support the dependence exhibited in trade durations because of the distributional similarity observed from our data with the help of fractal processes.

The paper is organized as follows. A brief review of point processes and several trade duration models based on estimating intensity of such processes is provided in Sect. 2. In Sect. 3, we introduce our empirical method based on trade duration data for 18 of the component stocks of the Dow Jones. We report our empirical results in Sect. 4. In this section, we compare the goodness of fit model used in our empirical study with the help of a testing procedure based on a bootstrap method we develop. We summarize our conclusions in Sect. 5.

2 Point processes in modeling durations

Given a probability space $(\Omega, \mathcal{A}, \mathrm{P})$, a family of random variables $(X_t)_{t\in\mathcal{T}}$ on Ω with values in some set M, (i.e., for all $t \in \mathcal{T}$ and $\mathcal{T}$ is some index set), $X_t : (\Omega, \mathcal{A}) \to (\mathrm{M}, \mathcal{B})$ is defined as a stochastic process with index set $\mathcal{T}$ and state space M. A sequence $(T_n)_{n\in N}$ of positive real random variables is a point process if $T_n(\omega) < T_{n+1}(\omega)$ for all $\omega \in \Omega$ and all $n \in N$; and $\lim_{n\to\infty} T_n(\omega) = \infty$, for all $\omega \in \Omega$. T_n is

[2] Roughness refers to an intricate, highly irregular appearance on all resolutions. It could be thought of as the synonym of irregularity (see Mandelbrot 1997). Davies and Hall (1999) discuss the surface roughness and point out that fractal methods partition the characteristics of surface roughness into two parts, “one of them scale invariant and the other governed by propeties of scale” (see Davies and Hall 1999, p. 5).

called the nth arrival time and $T_n = \sum_{i=1}^{n} \tau_i$; and $\tau_n = T_n - T_{n-1}$ (where $\tau_1 = T_1$) is called the nth *waiting time* (*duration*) for the *point process*.

A stochastic process $(N_t)_{t\in[0,\infty)}$ is a *counting process* if: $N_t : (\Omega, \mathcal{A}) \rightarrow (\mathrm{N}_0, \mathrm{P}(\mathrm{N}_0))$ for all $t \geq 0$, $N_0 \equiv 0$; $N_s(\omega) \leq N_t(\omega)$, for all $0 \leq s < t$ and all $\omega \in \Omega$; $\lim_{s\to t, s>t} N_s(\omega) = N_t(\omega)$, for all $t \geq 0$ and all $\omega \in \Omega$; $N_t(\omega) - \lim_{s\to t, s>t} N_s(\omega) \in (0, 1)$, for all $t > 0$ and all $\omega \in \Omega$; and $\lim_{t\to\infty} N_t(\omega) = \infty$, for all $\omega \in \Omega$. A point process $(T_n)_{n\in N}$ corresponds to a counting process $(N_t)_{t\in[0,\infty)}$ and vice versa, i.e.,

$$N_t(\omega) = |\{n \in N : T_n(\omega) \leq t\}|$$

for all $\omega \in \Omega$ and all $t \geq 0$. For all $\omega \in \Omega$ and all $n \in N$,

$$T_n(\omega) = \min\{t \geq 0 : N_t(\omega) = n\}$$

The mean value function of the counting process is $m(t) = E(N_t)$, for $t \geq 0$, and $m : [0, \infty) \rightarrow [0, \infty)$, $m(0) = 0$. m is an increasing and a right continuous function with $\lim_{t\to\infty} m(t) = \infty$. If the mean value function is differentiable at $t > 0$, then the first-order derivative is called *the intensity of the counting process*. Defining $\lambda(t) = \mathrm{d}m(t)/\mathrm{d}t$,

$$\lim_{\Delta t\to 0, \Delta t\neq 0} \frac{1}{|\Delta t|} P\left(|N_{t+\Delta t} - N_t| = 1\right) = \lambda(t)$$

and

$$\lim_{\Delta t\to 0, \Delta t\neq 0} \frac{1}{|\Delta t|} P\left(|N_{t+\Delta t} - N_t| \geq 2\right) = 0$$

From the viewpoint of the point processes literature (for example, Daley and Vere-Jones 2003), ultra-high frequency financial data can be described as marked point processes; that is, the state space M is a product space of $R^2 \otimes \mathcal{M}$ where $\mathcal{M}$ is the mark space. The ultra-high frequency transaction data contain two types of processes: time of transactions and events observed at the time of the transaction (see Engle 2000; Engle and Lunde 2003). Those events can be identified or described by marks, such as trade prices, posted bid and ask price, and volume. The amount of time between events is the duration. The intensity is used to characterize the point processes and is defined as the expected number of events per time increment considered as a function of time. In survival analysis, the intensity equals the hazard rate. For n durations, $d_1, d_2, \ldots, d_n$, which are sampled from a population with density function f and corresponding cumulative distribution function F, the survival function $S(t)$ is

$$S(t) = P[d_i > t] = 1 - F(t)$$

and the intensity or hazard rate $\lambda(t)$ is

$$\lambda(t) = \lim_{\Delta t\to 0} \frac{P[t < d_i \leq t + \Delta t \mid d_i > t]}{\Delta t}$$

The survival function and the density function can be obtained from the intensity,

$$\lambda(t) = \frac{f(t)}{S(t)} = \frac{-d\log(S(t))}{dt}$$

Several models have been proposed to model durations by estimating the intensity.[3] The favored models in the literature are the autoregressive conditional duration (ACD) model proposed by Engle and Russell (1998), the stochastic conditional duration (SCD) model by Bauwens and Veredas (2004), and the stochastic volatility duration (SVD) model by Ghysels et al. (2004). The ACD model expresses the conditional expectation of duration as a linear function of past durations and past conditional expectation. The disturbance is specified as an exponential distribution and as an extension of the Weibull distribution. The SCD model assumes that a latent variable drives the movement of durations. Then the expected durations in the SCD model are treated as the observed durations driven by a latent variable. The SVD model tries to capture the mean and variance of durations. The SCD and SVD models are mixed distributions models. The SCD model combines Weibull and gamma distributions while in the SVD model the durations are expressed as independently and exponentially distributed with a gamma heterogeneity. Extensions to these models have been suggested in the literature. Jasiak (1998) offers the fractional integrated ACD model, Gramming and Maurer (2000) replace the Weibull distribution by the Burr distribution, Bauwens and Giot (2000) propose the logarithmic ACD model, and Zhang et al. (2001) introduce the threshold ACD model. Bauwens and Giot (2003) propose an asymmetric ACD model; Feng et al. (2004) propose a linear non-Gaussian state-space version of the SCD model to capture the leverage effect of the expected durations. Simonsen (2007) extends the ACD model to examine the dependence between durations.

The ACD(m,n) model specified in Engle and Russell (1998) is

$$\begin{aligned} d_i &= \psi_i\, \varepsilon_i \\ \psi_i &= \omega + \sum_{j=0}^{m} \alpha_j\, d_{i-j} + \sum_{j=0}^{n} \beta_j\, \psi_{i-j} \end{aligned}$$

Bauwens and Giot (2000) give the logarithmic version of the ACD model as follows

$$d_i = e^{\psi_i}\, \varepsilon_i$$

Two possible specifications of conditional durations are

$$\psi_i = \omega + \sum_{j=0}^{m} \alpha_j \log d_{i-j} + \sum_{j=0}^{n} \beta_j\, \psi_{i-j}$$

[3] See recent reviews of Bauwens and Hautsch (2007) and Sun et al. (2007b).

and

$$\psi_i = \omega + \sum_{j=0}^{m} \alpha_j \log \varepsilon_{i-j} + \sum_{j=0}^{n} \beta_j \, \psi_{i-j}$$

Zhang et al. (2001) extend the conditional duration to a switching-regime version. Defining $L_q = [l_{q-1}, l_q)$, and $q = 1, 2, \ldots, Q$ for a positive integer Q, where $-\infty = l_0 < l_1 < \ldots < l_q = +\infty$ are the threshold values, d_i follows a q-regime threshold ACD (TACD(m,n)) model; that is

$$\psi_i = \omega^{(q)} + \sum_{j=0}^{m} \alpha_j^{(q)} \, d_{i-j} + \sum_{j=0}^{n} \beta_j^{(q)} \, \psi_{i-j}$$

For example, if there is a threshold value l_h and $0 < h < q$, the TACD(1,1) model can be expressed as following:

$$\psi_i = \begin{cases} \omega_1 + \alpha_1 \, d_{i-1} + \beta_1 \, \psi_{i-1} & \text{if } 0 < d_{i-1} \leq l_h \\ \omega_2 + \alpha_2 \, d_{i-1} + \beta_2 \, \psi_{i-1} & \text{if } l_h < d_{i-1} < \infty \end{cases}$$

The threshold l_h determines the regime boundaries. Fernandes and Gramming (2005) propose the nonparametric tests for ACD models and suggested the practical application for estimation of intraday volatility patterns.

The SCD model given by Bauwens and Veredas (2004) takes the following form:

$$d_i = \Psi_i \, \varepsilon_i$$

where

$$\Psi_i = e^{\psi_i}$$
$$\psi_i = \omega + \beta \, \psi_{i-1} + u_i$$

in which $|\beta| < 1$, denotes I_{i-1} the information set before d_i, $u_i | I_{i-1} \sim N(0, \sigma^2)$, $\varepsilon_i | I_{i-1}$ follows some distribution with positive support, and u_i is independent of $\varepsilon_j | I_{i-1}$ for any i and j.

Ghysels et al. (2004) proposed a SVD model by assuming that durations are independently and exponentially distributed with gamma heterogeneity. More explicitly, the model can be expressed as

$$d_i = \frac{U_i}{c V_i}$$

where U_i and V_i are two independent variables with exponential distribution and gamma(a, a) distribution. Then this expression can be transferred with suitable nonlinear transformations to the expression with Gaussian factors:

$$d_i = \frac{\Psi(1, \Phi(F_1))}{c\Psi(a, \Phi(F_2))} = \frac{H(1, F_1)}{cH(1, F_2)}$$

where F_1 and F_2 are i.i.d standard normal variables, Φ is the cdf of the standard normal distribution, and $\Psi(a, \cdot)$ is the quantile function of the gamma(a, a) distribution.

3 Empirical study

In this section, we report the empirical tests that investigate the goodness of fit of several candidate distributional assumptions in modeling roughness of trade duration.

3.1 The data

Ultra-high frequency data of 18 Dow Jones index component stocks based on NYSE trading for the year 2003 are examined.[4] The companies in the sample are listed in Table 1. Our sample is considerably larger than other studies that have investigated trade duration. Table 2 lists those studies and the stocks included in each one. Note that for the studies that include US stocks, IBM is included in 7 of 11 studies and because the sample size is small, IBM constitutes a major part of those studies. IBM is included in our study also.

The trade durations were calculated for regular trading hours (i.e., overnight trading was not considered). Consistent with Engle and Russell (1998) and Ghysels et al. (2004), open trades are deleted in order to avoid effects induced by the opening auction. Therefore trade durations only from 10:00 to 16:00 are considered. In our dataset, we observe many consecutive zero durations, which implies the existence of multiple transactions within a second. We aggregate these intra-second transactions within a second (see Engle and Russell 1998).

The panels in Fig. 1 illustrate several sampled trade duration series. These plots show data characteristics that are consistent with the data patterns reported in the literature. For the trade durations of each stock in our study, sample period runs were performed from January 4, 2003 to December 31, 2003. We will let N denote the length of the sample, sub-sample series that have been randomly selected by a moving window with length T $(1 \leq T \leq N)$. Replacement is allowed in the sampling. Stoev and Taqqu (2004) suggest that $2^{14} - 6,000 = 10,384$ is the optimal length for a fractional stable noise series to be simulated efficiently. Therefore, in the empirical

[4] The data from were provided by The Securities Industry Research Center of Asia-Pacific in Australia. The Dow Jones index consists of 30 stocks. The whole database we developed included data from 1996 to 2003 for stocks that remained in the index over the entire period. Only 18 stocks satisfied that requirement and we use the data of 2003 in this study.

Table 1 Statistical characteristics of trade duration in 2003 for 18 stocks

Stock	Size	Mean	Min	Max	$\text{Hurst}_{\tilde{u_t}}$	Hurst_{d_t}	$\alpha_{\tilde{u_t}}$	α_{d_t}
Alcoa Inc.	511817	10.8858	0	6194	0.6139	0.7202	1.3291	1.1022
American Express	819489	6.9582	0	6148	0.6379	0.5774	1.5673	1.3316
Caterpillar	571251	10.1333	0	6139	0.6932	0.6289	1.3475	1.0286
E.I.DuPont de Nemours	715515	7.8020	0	6114	0.6107	0.6379	1.3832	1.1930
Walt Disney	813090	6.8123	0	6134	0.6936	0.6943	1.3827	1.2792
Eastman Kodak Co.	475512	11.9756	0	6145	0.6445	0.7108	1.4295	1.1715
General Electric	1188851	4.8295	0	6196	0.4814	0.4725	1.4833	1.4413
General Motors	684082	8.4509	0	6147	0.4647	0.5679	1.2921	1.1606
IBM	986153	5.8425	0	6143	0.5186	0.4952	1.6492	1.2792
Int.Paper Company	606742	9.1403	0	6174	0.6195	0.7110	1.3677	1.0742
Coca-Cola Co.	695948	8.1239	0	6145	0.5676	0.5763	1.3470	1.1606
McDonalds	615302	9.0091	0	6165	0.5548	0.6555	1.3144	1.2437
3M Co.	739258	7.7422	0	6142	0.5819	0.5948	1.3705	1.2349
Altria Group	846140	6.7920	0	6149	0.5396	0.5842	1.3825	1.1282
Merck & Co.	875457	6.6009	0	6135	0.5861	0.5252	1.4886	1.1715
Procter & Gamble	780633	7.2657	0	6166	0.6853	0.5704	1.4437	1.2792
AT&T Inc.	553896	10.1818	0	6161	0.6280	0.6894	1.4133	1.1542
United Technologies	657286	8.5779	0	6161	0.6051	0.6544	1.3176	1.1282

analysis, sub-sample length (i.e., the window length) of $T = 10,384$ was chosen. A total of 684 sub-samples were randomly created.

3.2 The methodology of finding the best model

In our empirical study, we simulate a series for each distributional assumption with and without subordinating them into the ACD(1,1) structure. Then we compare the goodness of fit of our model.

The ACD model is defined as follows:

$$d_i = \psi_i u_i,$$

and

$$\psi_i = \kappa + \sum_{t=1}^{p} \gamma_i d_{i-t} + \sum_{j=1}^{q} \theta_j \psi_{i-j},$$

u_i can be calculated from d_i/ψ_i. We define

$$\tilde{u_i} = \frac{d_i}{\hat{\psi}_i},$$

Table 2 Studies of trade duration and stocks included in sample

Study	Exchange	Stocks	Model distribution(s)
Bauwens (2005)	Tokyo Stock Exchange	Nippon Steel, Sony, Tokyo Electric, Toyota	Generalized gamma
Bauwens and Veredas (2004)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon	Weibull
Bauwens and Giot (2000)	New York Stock Exchange	Boeing, Walt Disney, IBM	Exponential, Weibull
Bauwens and Giot (2003)	New York Stock Exchange	Walt Disney, IBM	Weibull
Bauwens and Veredas (2004)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon	Burr, Exponential, Generalized gamma, Weibull
Engle and Russell (1998)	New York Stock Exchange	IBM	Exponential, Weibull
Engle and Lunde (2003)	New York Stock Exchange	Bank-American, Walt Disney, General Motors Federal National Mortagage, McDonald's, Monsanto, Procter & Gamble, Schlumberger	Exponential
Feng et al. (2004)	New York Stock Exchange	Boeing, Coca Cola, IBM	Exponential, log- Weibull, log-Gamma
Fernandes and Gramming (2005)	New York Stock Exchange	Exxon	Exponential, Weibull, Burr, Generalized gamma
Fernandes and Gramming (2006)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon, IBM	Burr
Ghysels et al. (2004)	Paris Stock Exchange	Alctel	Exponential, Gamma
Jasiak (1998)	Paris Stock Exchange	Alctel	Weibull
	New York Stock Exchange	IBM	
Zhang et al. (2001)	New York Stock Exchange	IBM	Generalized gamma

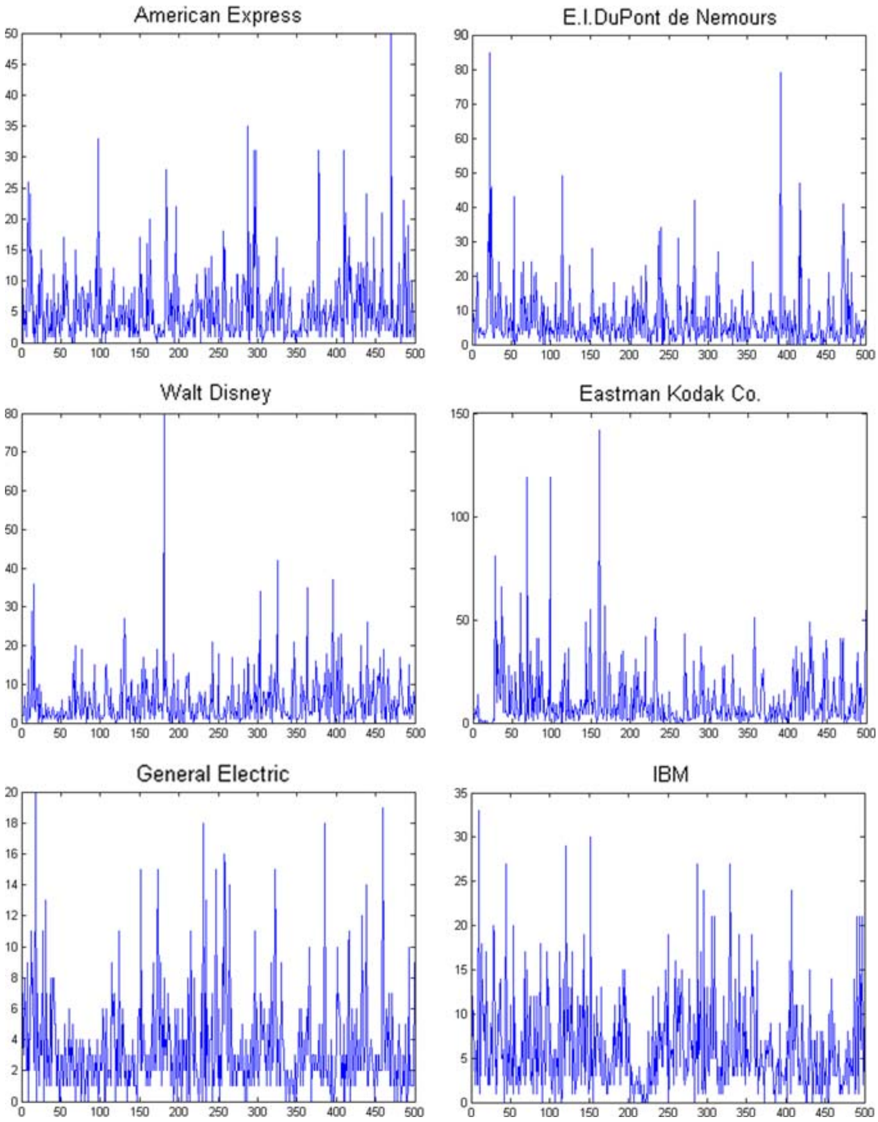

Fig. 1 Plot of trade duration for several stocks

where $\hat{\psi}_i$ is the estimation of ψ_i. In our empirical analysis, an ACD(1,1) model structure is adopted. The objective is to check the statistical characteristics exhibited by trade duration d_i and the error term $\tilde{u}_i$ in ACD(1,1) structure. We simulate d_i and $\tilde{u}_i$ with the ACD(1,1) structure based on the parameters estimated from the empirical series. Then we test the goodness of fit between the empirical series and the simulated series. Six candidate distributional assumptions—lognormal distribution, stable distribution, exponential distribution, Weibull distribution, fractional Gaussian noise, and

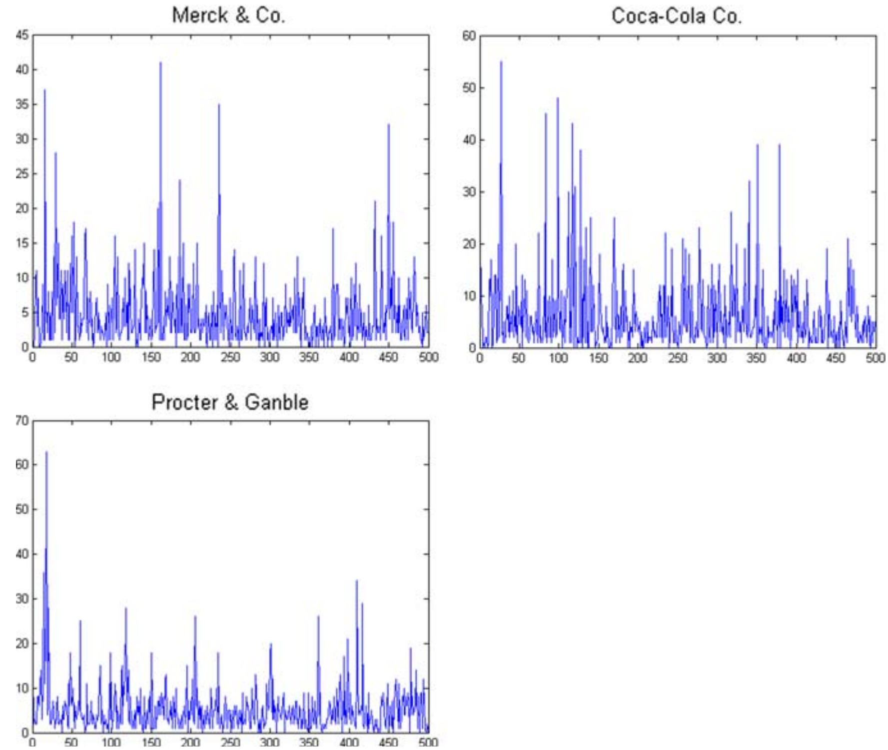

Fig. 1 continued

fractional stable noise[5]—are analyzed for estimation, simulation, and testing. Trade durations are positive numbers, therefore the stable distribution, fractional Gaussian noise, and fractional stable noise are defined on positive supports correspondingly.

4 Results

4.1 Preliminary tests

The descriptive statistics of the trade duration data in our study are presented in Table 1. The second column shows the number of observations for each stock in our study. The mean, maximum, and minimum of trade duration for each stock are shown in Table 1. In our dataset, General Electric and IBM are the most frequently traded stocks. It can be seen that trade durations exhibit diurnal patterns, especially, long duration can be observed at lunch time (see Engle 2000). The maximum value of trade duration

[5] The specifications of stable distribution and fractal processes are shown in the Appendix. Further information about the stable distribution and fractal processes can be found in Rachev and Mittnik (2000), Sun et al. (2007b) and references therein.

is reported in the fifth column in Table 1. In our dataset, we removed the overnight durations.

The Hurst index $H \in (0, 1)$ usually serves as the measure of the tendency of a process and stands for the self-similarity index in Gaussian stochastic processes. It can be somewhat explained by considering the covariance of two consecutive increments. When $H \in (0, 0.5)$, the increments of a process tend to have opposite signs and thus are more zigzagging due to the negative covariance; when $H \in (0.5, 1)$, the covariance between these two increments is positive and less zigzagging of the process; when $H = 0.5$, the covariance between this two increments is zero. It can be stated as follows: If the Hurst index is less than 0.5, the process displays "anti-persistence" which means that the positive excess return is more likely to be reversed and the performance in the next period is likely to be below the average, or on the contrary, the negative excess return is more likely to be reversed and the performance in the next period is likely to be above the average. If the Hurst index is greater than 0.5, the process displays "persistence" which means that the positive excess return or the negative excess return is more likely to be continued and the performance in the next period is likely to be the same as that in the current period. If the Hurst index is equal to 0.5, the process displays no memory, which means the performance in the next period has equal probability to be below and above the performance in the current period. From Table 1, we find that the Hurst index has no value of 0.5, which indicates that the memory effect occurs in our samples.

The Hurst index for non-Gaussian stable processes has different bounds for "persistence" and "anti-persistence". For tail index $\alpha \in (0, 2)$, when $H \in (0, 1/\alpha)$, the processes exhibit "anti-persistence", and when $H \in (1/\alpha, 1)$, the processes exhibit "persistence". There is no long-range dependence when $\alpha \in (0, 1]$ because the Hurst index is bounded in the interval $(0, 1)$. When $H = 1/\alpha$, depending on the value of α the processes exhibit either no memory or long-range dependence.[6] From Table 1, we find that the Hurst index has no value of $1/\alpha$. Therefore, we find that long-range dependence occurs in our samples. This result reveals the phenomenon that when there is information in the market, speed of trading is faster and fast trades are close together. Since trade duration can reflect the information flows in the market, we can explain this result as follows. When there is information, typically there exists a higher proportion of information traders. These traders trade with higher speed and fast trades cluster together. In contrast, when there is less information, the speed of trades slows down till the new information is captured. (The empirical evidence can be found in Dufour and Engle (2000), Engle and Russell (1998), and Engle (2000).)

Our results indicate that the Hurst index has no value of 0.5 or $1/\alpha$. As described by Mandelbrot (2005), such patterns are the "wild" forms of roughness which require considering fractal models. The results reported in Table 1 confirm Mandelbrot's statement that "the proper language of the theory of roughness in nature and culture is fractal geometry" (see Mandelbrot 2005, p. 193).

We use the Ljung–Box–Pierce Q-statistic based on autocorrelation function to test the serial correlation (i.e., the memory effect). The Q-statistic is given as follows:

[6] For a detailed discussion see Samorodnitsky and Taqqu (1994) and Cohen and Samorodnitsky (2006).

$$Q :\sim \chi_m^2 = N(N+2) \sum_{k=1}^{m} \frac{\rho_k^2}{N-k},$$

where, N denotes the sample size, m the number of autocorrelation lags included in the statistic, and ρ_k the sample autocorrelation at lag order k which is

$$\rho_k = \frac{\sum_{t=1}^{N-k} d_i \, d_{i+k}}{\sum_{t=1}^{N} d_i^2}.$$

The Q-statistic follows the asymptotic chi-square distribution with m degrees of freedom.

The null hypothesis that there is no serial correlation can be rejected at different lags based on the results reported in Table 3. These results are similar to the empirical results in Simonsen (2007). This table implies that the memory effect occurs in each duration series. In order to see when the memory effect vanishes, we compare the Q-statistic with its corresponding critical value. When the quotation of the Q-statistic and the corresponding critical value are less than 1, we cannot reject the null hypothesis that there is no serial correlation. We also show such quotations in Table 3. From this table, all the trade durations exhibit serial correlation. After 500 lags, the memory effect vanishes for 7 stocks and after 1,500 lags, the memory effect vanishes for 13 stocks. From the ratios in Table 3, we find that the speed of autocorrelation decay is declining, which confirms the effect of long-range dependence.

4.2 Goodness of fit test

The Kolmogorov–Simirnov distance (KS) and Anderson–Darling distance (AD) proposed by Rachev and Mittnik (2000) are used as the criterion for the goodness of fit testing. They are defined as following:

$$\text{KS} = \sup_{x \in \Re} \left| F_s(x) - \tilde{F}(x) \right|,$$

and

$$\text{AD} = \sup_{x \in \Re} \frac{\left| F_s(x) - \tilde{F}(x) \right|}{\sqrt{\tilde{F}(x)(1 - \tilde{F}(x))}},$$

where $F_s(x)$ denotes the empirical sample distribution and $\tilde{F}(x)$ is the estimated distribution function. The major disadvantage of KS statistic researchers have argued is that it tends to be more sensitive near the center of the distribution than at the tails. The AD statistic can overcome this. The reliability of testing the empirical distribution will be increased with the help of these two statistics, with the KS distance focusing on the deviations around the median of the distribution and the AD distance on the discrepancies in the tails.

The AD and KS statistics are calculated for the six candidate distributional assumptions. Table 4 reports the descriptive statistics of the computed AD and KS statistics.

Table 3 Ljung–Box–Pierce Q-test statistic for different lags at α=0.05

Stock	lag10	lag20	lag50	lag100	lag200	lag500	lag1000	lag1500
Alcoa Inc.	760.3	931.7	1379.5	2041.0	3224.5	5912.9	9897.8	13327.0
	41.5	*29.7*	*20.4*	*16.4*	*13.8*	*10.7*	*9.2*	*8.4*
American Express	330.5	336.2	344.5	352.8	372.8	405.6	596.7	691.3
	18.1	*10.7*	*5.1*	*2.8*	*1.6*	*0.7*	*0.6*	*0.4*
Caterpillar	413.9	446.1	516.6	623.5	814.5	1329.9	2093.9	2899.9
	22.6	*14.2*	*7.6*	*5.0*	*3.4*	*2.4*	*1.9*	*1.8*
E.I.DuPont de Nemours	371.6	401.4	455.6	527.1	644.4	840.9	1184.5	1376.5
	20.3	*12.7*	*6.7*	*4.2*	*2.7*	*1.5*	*1.1*	*0.8*
Walt Disney	294.5	319.2	374.8	452.1	606.5	936.2	1475.7	1620.3
	16.1	*10.1*	*5.5*	*3.6*	*2.5*	*1.6*	*1.3*	*1.0*
Eastman Kodak Co.	157.9	177.6	204.4	257.0	351.2	654.6	1078.8	1530.8
	8.6	*5.6*	*3.0*	*2.0*	*1.5*	*1.1*	*1.1*	*0.9*
General Electric	386.8	391.1	393.1	394.8	397.4	406.6	601.1	627.5
	21.1	*12.4*	*5.8*	*3.1*	*1.6*	*0.7*	*0.5*	*0.3*
General Motors	350.8	356.2	365.7	373.3	396.8	513.5	584.7	832.1
	19.1	*11.3*	*5.4*	*3.0*	*1.6*	*0.9*	*0.5*	*0.5*
IBM	159.9	165.7	169.7	174.0	181.2	195.7	357.1	418.0
	8.7	*5.2*	*2.5*	*1.3*	*0.7*	*0.3*	*0.3*	*0.2*
Int. Paper Company	344.3	400.2	519.7	698.0	986.6	1424.1	1846.4	2299.0
	18.8	*12.7*	*7.6*	*5.6*	*4.2*	*2.5*	*1.7*	*1.4*
Coca-Cola Co.	244.2	253.4	266.0	293.4	318.8	366.1	620.3	758.9
	13.3	*8.0*	*3.9*	*2.3*	*1.3*	*0.6*	*0.5*	*0.4*
McDonalds	233.2	256.1	292.2	346.9	442.6	899.3	1016.4	1343.8
	12.7	*8.1*	*4.3*	*2.7*	*1.8*	*1.6*	*0.9*	*0.8*
3M Co.	515.3	528.2	551.2	579.0	631.5	750.7	1044.4	1340.2
	28.1	*16.8*	*8.1*	*4.6*	*2.6*	*1.3*	*0.9*	*0.8*
Altria Group	279.1	284.2	292.6	299.0	319.1	357.7	556.0	617.1
	15.2	*9.0*	*4.3*	*2.4*	*1.3*	*0.6*	*0.5*	*0.3*
Merck & Co.	262.9	268.0	276.1	280.0	285.9	303.0	461.6	535.6
	14.3	*8.5*	*4.0*	*2.2*	*1.2*	*0.5*	*0.4*	*0.3*
Procter & Gamble	517.6	522.9	533.5	541.1	553.0	580.6	774.7	824.3
	28.2	*16.6*	*7.9*	*4.3*	*2.3*	*1.1*	*0.7*	*0.5*
AT&T Inc.	280.2	319.1	398.1	517.9	691.1	990.2	1370.2	1709.4
	15.3	*10.1*	*5.8*	*4.1*	*2.9*	*1.7*	*1.2*	*1.1*
United Technologies	327.1	356.2	417.2	491.2	621.1	834.9	1130.4	1389.9
	17.8	*11.3*	*6.1*	*3.9*	*2.6*	*1.5*	*1.1*	*0.8*
Critical Value	18.3	31.4	67.5	124.3	233.9	553.1	1074.7	1591.2

The italic numbers show the ratios of Q-test statistics compared with corresponding critical values

Table 4 Summary of KS statistics for alternative distributional assumptions, " * " indicates the test for d_t, otherwise for $\tilde{u_t}$. Mean, median, standard deviation ("std"), maximum value ("max"), minimum value ("min") and range of AD and KS statistics are presented in this table

	AD_{mean}	AD_{median}	AD_{std}	AD_{max}	AD_{min}	AD_{range}	AD^*_{mean}	AD^*_{median}	AD^*_{std}	AD^*_{max}	AD^*_{min}	AD^*_{range}
FGN	2.7056	1.7220	2.7171	15.2900	0.1732	15.1170	4.4772	3.9688	3.0290	19.3230	0.2423	19.0810
fsn	0.4468	0.4207	0.1809	1.1384	0.1066	1.0318	0.4574	0.4307	0.1836	1.2071	0.1014	1.1057
lognormal	2.6100	1.6285	2.7004	15.4960	0.1652	15.3310	4.3202	3.8918	3.0646	19.6470	0.2407	19.4070
stable	0.4460	0.4233	0.1787	1.1384	0.1122	1.0262	0.4583	0.4372	0.1835	1.0795	0.1060	0.9736
exponential	1.0553	1.0516	0.0198	1.1521	0.9719	0.1802	1.0572	1.0526	0.0174	1.1246	1.0254	0.0993
Weibull	1.0487	1.0472	0.0220	1.1103	0.9471	0.1632	1.0572	1.0526	0.0174	1.1246	1.0254	0.0993
	KS_{mean}	KS_{median}	KS_{std}	KS_{max}	KS_{min}	KS_{range}	KS^*_{mean}	KS^*_{median}	KS^*_{std}	KS^*_{max}	KS^*_{min}	KS^*_{range}
FGN	0.2615	0.2901	0.0866	0.3955	0.0644	0.3311	0.3157	0.3359	0.0715	0.4054	0.0902	0.3152
fsn	0.0381	0.0361	0.0101	0.0887	0.0197	0.0690	0.0493	0.0484	0.0097	0.0920	0.0288	0.0632
lognormal	0.2590	0.2816	0.0858	0.3984	0.0613	0.3371	0.3129	0.3334	0.0717	0.4030	0.0891	0.3139
stable	0.0378	0.0366	0.0099	0.0861	0.0195	0.0666	0.0495	0.0484	0.0101	0.0923	0.0281	0.0641
exponential	0.5265	0.5249	0.0091	0.5579	0.4856	0.0723	0.5278	0.5257	0.0081	0.5586	0.5126	0.0459
Weibull	0.5236	0.5230	0.0105	0.5522	0.4729	0.0793	0.5278	0.5257	0.0081	0.5586	0.5126	0.0459

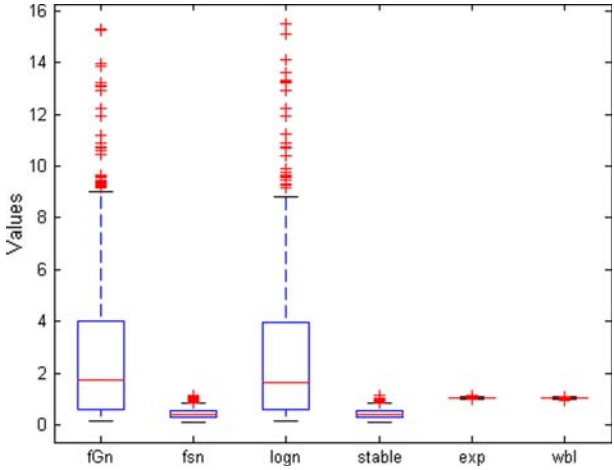

Fig. 2 Boxplot of AD statistics for $\tilde{u_t}$ in alternative distributional assumptions

From Table 4, fractional stable noise and stable distribution exhibit a smaller mean value for the AD and KS statistics in comparison with the other four distributions. Figure 2 shows the boxplot of AD statistics of $\tilde{u_t}$ for the six alternative distributional assumptions investigated. Figure 3 shows the boxplot of AD statistics for d_t. Figure 4 shows the boxplot of KS statistics of $\tilde{u_t}$ for the six alternative distributional assumptions. Figure 5 shows the boxplot of KS statistics of d_t. These figures show that fractional stable noise and stable distribution have a small value of AD and KS statistics, confirming the results reported in Table 4. These results indicate that with or without an ACD(1,1) model structure, the fractional stable noise and stable distribution perform better than the other four tested distributional assumptions based on the criterion for goodness of fit testing.

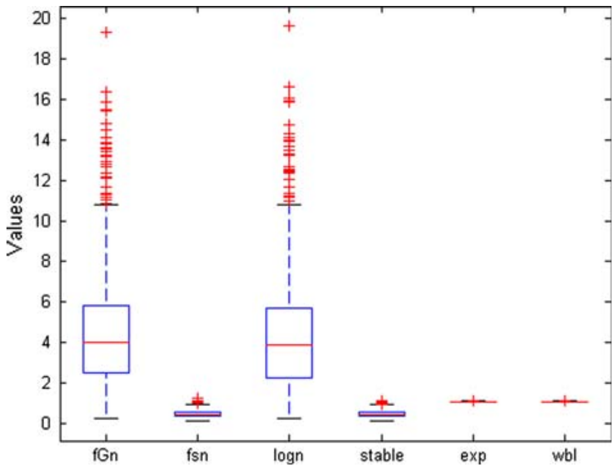

Fig. 3 Boxplot of AD* statistics for d_t in alternative distributional assumptions

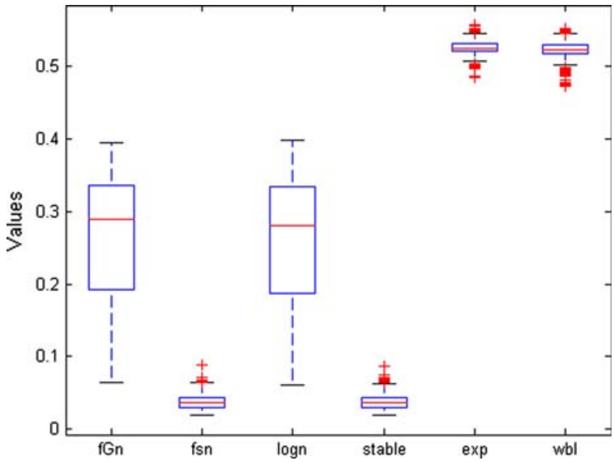

Fig. 4 Boxplot of KS statistics for $\tilde{u_t}$ in alternative distributional assumptions

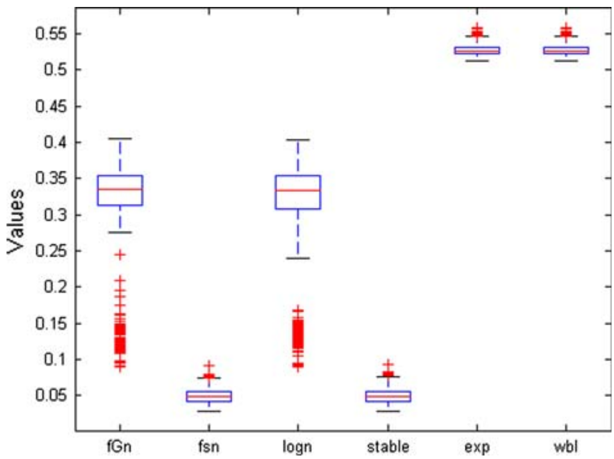

Fig. 5 Boxplot of KS* statistics for d_t in alternative distributional assumptions

From Figs. 2, 3, 4 and 5, we can see that the fractional stable noise and the stable distribution fit $\tilde{u_t}$ and d_t better than other distributional assumptions. In order to empirically examine our conjecture, we formulate a statistical test procedure. Because we know that smaller AD and KS statistics mean better goodness of fit, in our test we are going to statistically test how significantly "smaller" AD and KS statistics are. The hypothesis test is:

$$H_0 : \mu_{\text{criterion 1}} - \mu_{\text{criterion 2}} \geq 0$$
$$H_1 : \mu_{\text{criterion 1}} - \mu_{\text{criterion 2}} < 0$$

where $\mu_{\text{criterion}}$ is the mean value of AD or KS statistics of the candidate distributional assumptions investigated. The distributions of AD and KS values are unknown. All

Table 5 Bootstrap 99% confidence intervals for mean of differences in AD and KS statistics, " * " indicates statistics for d_t, otherwise for $\tilde{u_t}$. "FGN" stands for fractional Gaussian noise,"fsn" stands for fractional stable noise,"exp" stands for exponential distribution, "wbl" stands for Weibull distribution

T:	AD	AD*	KS	KS*
$E(T_{\text{stable}} - T_{\text{FGN}})$	(−2.5223, −1.9851)	(−4.3235, −3.7124)	(−0.2318, −0.2157)	(−0.2734, −0.2594)
$E(T_{\text{stable}} - T_{\text{lognormal}})$	(−2.4281, −1.8893)	(−4.1642, −3.5481)	(−0.2292, −0.2132)	(−0.2706, −0.2565)
$E(T_{\text{stable}} - T_{\text{exp}})$	(−0.6278, −0.5913)	(−0.6175, −0.5803)	(−0.4895, −0.4878)	(−0.4793, −0.4774)
$E(T_{\text{stable}} - T_{\text{wbl}})$	(−0.6211, −0.5850)	(−0.6175, −0.5804)	(−0.4868, −0.4848)	(−0.4793, −0.4773)
$E(T_{\text{fsn}} - T_{\text{FGN}})$	(−2.5236, −1.9774)	(−4.3204, −3.7077)	(−0.2315, −0.2154)	(−0.2736, −0.2597)
$E(T_{\text{fsn}} - T_{\text{lognormal}})$	(−2.4265, −1.8888)	(−4.1638, −3.5485)	(−0.2290, −0.2130)	(−0.2708, −0.2568)
$E(T_{\text{fsn}} - T_{\text{exp}})$	(−0.6274, −0.5904)	(−0.6182, −0.5815)	(−0.4893, −0.4875)	(−0.4795, −0.4776)
$E(T_{\text{fsn}} - T_{\text{wbl}})$	(−0.6203, −0.5838)	(−0.6185, −0.5815)	(−0.4866, −0.4845)	(−0.4795, −0.4776)
$E(T_{\text{fsn}} - T_{\text{stable}})$	(−0.0057 , 0.0074)	(−0.0077 , 0.0059)	(−0.0002 , 0.0007)	(−0.0007 , 0.0002)

AD or KS values are expressed as i.i.d. random variables $X_1, X_2, \ldots, X_n$, each with distribution function $F_X(\cdot|\theta)$. A $100(1-\alpha)\%$ upper confidence bound (UCB) is defined as $U(X_1, X_2, \ldots, X_n)$ for a function of $h(\theta)$ if for every θ,

$$P_\theta\left(h(\theta) \leq U(X_1, X_2, \ldots, X_n)\right) \geq 1-\alpha$$

and $(-\infty, U(X_1, X_2, \ldots, X_n)]$ is the $100(1-\alpha)\%$ upper confidence interval for $h(\theta)$. Similarly, $L(X_1, X_2, \ldots, X_n)$ is a $100(1-\alpha)\%$ lower confidence bound (LCB) for the function $h(\theta)$ for every θ

$$P_\theta\left(h(\theta) \geq L(X_1, X_2, \ldots, X_n)\right) \geq 1-\alpha$$

and $[L(X_1, X_2, \ldots, X_n), +\infty)$ is the $100(1-\alpha)\%$ lower confidence interval for $h(\theta)$.

As hypothesis testing and confidence intervals are dual concepts, the hypothesis testing is in fact evaluated by following test rules; that is, (1) if UCB is less than zero, H_0 can be rejected, (2) if LCB is greater than zero, H_0 cannot be rejected, and (3) if H_0 is greater than LCB but at the same time less than UCB, there is no statistically significant conclusion. Employing the bootstrap method introduced in DiCiccio and Efron (1996), the 99% bootstrap confidence intervals are reported in Table 5. From this table, at a high confidence level, the fractional stable noise and stable distribution are more suitable to modeling trade duration data with or without support of an ACD(1,1) structure.

In comparing the fractional stable noise and stable distribution, it is unclear as to whether the fractional stable noise is better than the stable distribution or vice versa. We compare the supporting cases for the fractional stable noise and stable distribution. We present our comparison results in Table 6.

The fractional stable noise has a slightly better support than stable distribution as a single process in modeling trade durations. Since the fractional stable noise belongs to fractal processes, this result confirms the superiority of fractal processes in modeling roughness exhibited in trade duration (see Mandelbrot 2005 and references therein).

Table 6 Supporting cases comparison of goodness of fit for fractional stable noise and stable distribution based on AD and KS statistics

	AD	AD*	KS	KS*
fsn $\succ$ stable	327	345	327	362
	(47.81%)	(50.44%)	(47.81%)	(52.93%)
stable $\succ$ fsn	344	328	351	318
	(50.29%)	(47.95%)	(51.32 %)	(46.49%)
fsn $\sim$ stable	13	11	6	4
	(1.90%)	(1.61 %)	(0.87%)	(0.58%)

Symbol " * " indicates the test for d_t, otherwise the test is for $\tilde{u_t}$. Symbol " $\succ$" means being preferred and "$\sim$" means indifference. Numbers shows the supporting cases to the statement in the first column and the number in parentheses give the proportion of supporting cases in the whole sample

The stable distribution has a greater number of supporting cases in comparison to the fractional stable noise in modeling duration data with an ACD(1,1) structure. The stable distribution has advantages in capturing heavy-tailedness in observations. As we mentioned above, trade durations reflect the information flows in the market. When there is information in the market, a higher proportion of information traders trade in the market with a higher speed of trading. Similarly, when there is less or no information in the market, a higher proportion of traders trade in the market with a lower speed of trading. Therefore, extreme events (i.e., very short durations and very long durations) are often observed in the market, causing the distribution of durations to have heavy-tails. The ACD model with a stable distribution captures simultaneously both extreme events and clusters of extreme events driven by information in the market.

Several researchers find evidence that fractal processes or the stable distribution are better in modeling market volatility (see Rachev and Mittnik 2000; Sun et al. 2007a). As we know, price adjustments are closely connected with information revealed from the market. When there is information, the proportion of information traders tends to be higher and the speed of price adjustment tends to be faster. Extreme events can be observed both from trade durations and price adjustment. In our study, we show that the fractional stable noise and stable distribution can capture the trade duration movements well. This result explains why fractional stable noise and stable distribution exhibit superiority in modeling price adjustments because of the tight relationship between the speed of price adjustment and information revealed in trade duration. Our models capture the situation of fast price adjustments and trade in clusters.

In addition, by analyzing a large sample as we have done in our study, we can detect evidence that some trade durations of different stocks exhibit similar movements, see Fig. 1. For example, the panels of Eastman Kodak Co. and Merck & Co. and the panels of General Electric and IBM exihibit similar co-movement patterns. The findings of Simonsen (2007) reveal that there is a certain dependence in trade durations of stocks. Our empirical results partially support that finding. In our study, most cases provide support that the ACD model with stable distribution is preferred. In our study, the trade duration dynamics exhibit a movement/co-movement which can be captured by the same model. This might lead us to find further evidence of the dependence in our sample with relatively robust tests.

5 Conclusions

The empirical research with very few stocks have demonstrated that trade duration data exhibit three characteristics: long-range dependence, heavy tailedness, and clustering. In this paper, we investigate the presence of these characteristics using a larger number of stocks and investigate whether for modeling trade duration data: (1) a single stochastic process capturing long-range dependence and heavy tailedness and (2) a relatively powerful distributional assumption in a relatively simple functional structure can be used.

To examine these issues, we introduce fractional stable noise and fractional Gaussian noise to capture long-range dependence and heavy tailedness in modeling trade duration. In our empirical analysis, we investigate six distributional assumptions (fractional stable noise, fractional Gaussian noise, stable distribution, lognormal distribution, exponential distribution, and Weibull distribution) for modeling the trade duration for 18 Dow Jones index component stocks. By using parameters estimated from the empirical series, we simulate a series for each distributional assumption with and without subordinating them into an ACD(1,1) structure. Then we compare the goodness of fit for these generated series to the empirical series by adopting two test criteria for testing heavy-tailed distributions, the Kolmogorov-Simirnov and Anderson-Darling statistics. A test procedure is formulated based on a bootstrap method, and it is used in order to obtain empirical results.

The above test procedure yields empirical evidence which shows that the stable distribution and fractional stable noise are better in modeling trade duration than the exponential distribution, lognormal distribution, Weibull distribution, and fractional Gaussian noise. The results indicate that residuals from the ACD(1,1) model are more likely to be described by a stable distribution and trade durations exhibit the features of fractional stable noise. That is, stable distribution subordinated with an ACD(1,1) structure and fractional stable noise, demonstrate superior performance in the modeling of trade duration.

Our results are consistent with the general findings in the literature on trade duration: informative and short trade durations move prices more than long trade duration. Based on the models we employ, our findings also explain the relationship between price adjustment and information revealed in trade duration. In addition, our results confirm the advantage of fractal models in the study of roughness in trade duration and partially provide evidence for duration dependence.

We argue that it is critical that the findings reported in this paper be taken into account in modeling trade duration. Many studies have found that the stable distribution is a better description of financial data because it can capture heavy tailedness and has a close relationship with long-range dependence. As a self-similar process, fractional stable noise can capture almost all reported stylized facts in financial return data, such as heavy tailedness, long memory, non-Gaussian characters, and clustering. Therefore, if the fractional stable noise and stable distribution can be properly employed in financial modeling, more accurate prediction might be realized by well-defined functional models.

Appendix

Specification of the self-similar processes

Lamperti (1962) first introduced the semi-stable processes (which we today refer to as self-similar processes). Let T be either R, $R_+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$. The real-valued process $\{X(t), t \in T\}$ has stationary increments if $X(t + a) - X(a)$ has the same finite-dimensional distrituions for all $a \geq 0$ and $t \geq 0$. Then the real-valued process $\{X(t), t \in T\}$ is self-similar with exponent of self-similarity H for any $a > 0$, and $d \geq 1, t_1, t_2, \ldots, t_d \in T$, satisfying:

$$\Big(X(at_1), X(at_2), \ldots, X(at_d)\Big) \overset{d}{=} \Big(a^H X(t_1), a^H X(t_2), \ldots a^H X(t_d)\Big). \tag{1}$$

Fractional Gaussian noise

Kolmogoroff (1940) first introduced the fractional Brownian motion (FBM) and Samorodnitsky and Taqqu (1994) clarify the definition of FBM as a Gaussian process having self-similarity index H and stationary increments. Mandelbrot and Van Ness (1968) defined the stochastic representation

$$B_H(t) := \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]dB(s) + \int_{0}^{t}(t-s)^{H-\frac{1}{2}}dB(s)\right),$$

where $\Gamma(\cdot)$ represents the Gamma function:

$$\Gamma(a) := \int_{0}^{\infty} x^{a-1}e^{-x}dx,$$

and $0 < H < 1$ is the Hurst parameter. The integrator B is the ordinary Brownian motion. The main difference between fractional Brownian motion and ordinary Brownian motion is that the increments in Brownian motion are independent while in fractional Brownian motion they are dependent. As to the fractional Brownian motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional Gaussian noise (FGN), which is, for $j = 0, \pm1, \pm2, \ldots$, $Y_j = B_H(j-1) - B_H(j)$.

Fractional stable noise

Fractional Brownian motion can capture the effect of long-range dependence, but with less power to capture the heavy tailedness. The existence of abrupt discontinuities in financial data, combined with the empirical observation of sample excess kurtosis and unstable variance, confirms the stable Paretian hypothesis in Mandelbrot (1963, 1997). It is natural to introduce the stable Paretian distribution in self-similar processes in order to capture both long-range dependence and heavy tailedness. Samorodnitsky and Taqqu (1994) introduce the α-stable H-*sssi* processes $\{X(t), t \in R\}$ with $0 < \alpha < 2$. If $0 < \alpha < 1$, the values of the Hurst parameter are $H \in (0, 1/\alpha]$ and if $1 < \alpha < 2$, the values of the Hurst parameter are $H \in (0, 1]$. There are many different extensions of fractional Brownian motion to the stable distribution. The most commonly used is the linear fractional stable motion (also called linear fractional Lévy motion), $\{L_{\alpha,H}(a, b; t), t \in (-\infty, \infty)\}$, which is defined by Samorodnitsky and Taqqu (1994) as follows:

$$L_{\alpha,H}(a, b; t) := \int_{-\infty}^{\infty} f_{\alpha,H}(a, b; t, x) M(dx), \tag{2}$$

where

$$f_{\alpha,H}(a, b; t, x) := a\left((t-x)_{+}^{H-\frac{1}{\alpha}} - (-x)_{+}^{H-\frac{1}{\alpha}}\right) + b\left((t-x)_{-}^{H-\frac{1}{\alpha}} - (-x)_{-}^{H-\frac{1}{\alpha}}\right), \tag{3}$$

and where a, b are real constants, $|a| + |b| > 1, 0 < \alpha < 2, 0 < H < 1, H \neq 1/\alpha$ and M is an α-stable random measure on R with Lebesgue control measure and skewness intensity $\beta(x), x \in (-\infty, \infty)$ satisfying: $\beta(\cdot) = 0$ if $\alpha = 1$. Samorodnitsky and Taqqu (1994) define linear fractional stable noise expressed by $Y(t)$, and $Y(t) = X_t - X_{t-1}$,

$$\begin{aligned} Y(t) &= L_{\alpha,H}(a, b; t) - L_{\alpha,H}(a, b; t-1) \\ &= \int_R \left(a\left[(t-x)_{+}^{H-\frac{1}{\alpha}} - (t-1-x)_{+}^{H-\frac{1}{\alpha}} \right] \right. \\ &\quad \left. + b\left[(t-x)_{-}^{H-\frac{1}{\alpha}} - (t-1-x)_{-}^{H-\frac{1}{\alpha}} \right] \right) M(dx) \end{aligned}$$

where $L_{\alpha,H}(a, b; t)$ is linear fractional stable motion defined by Eqs. (1) and (2), and M is a stable random measure with Lebesgue control measure given $0 < \alpha < 2$.[7] In this paper, if there is no special indication, fractional stable noise (fsn) is generated from linear fractional stable motion.

Stable Paretian distribution

Stable distribution requires four parameters for complete description: an index of stability $\alpha \in (0, 2]$ (also called the tail index), a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$, and a location parameter $\zeta \in \Re$. There is unfortunately no closed-form expression for the density function and distribution function of a stable

[7] Some properties of these processes have been discussed in Mandelbrot and Van Ness (1968), Maejima and Rachev (1987), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Samorodnitsky (1994), and Samorodnitsky and Taqqu (1994). Stoev et al. (2002) and Stoev and Taqqu (2004) discuss the method of estimation and simulation for the fractional stable motion.

distribution. Rachev and Mittnik (2000) give the definition of the stable distribution: A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma \geq 0$ and ζ real such that its characteristic function has the following form:

$$E\exp(i\theta X) = \begin{cases} \exp\{-\gamma^{\alpha}|\theta|^{\alpha}(1 - i\beta(\sin\theta)\tan\frac{\pi\alpha}{2}) + i\zeta\theta\} & \text{if } \alpha \neq 1 \\ \exp\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\sin\theta)\ln|\theta|) + i\zeta\theta\} & \text{if } \alpha = 1 \end{cases}$$

and,

$$\operatorname{sign}\theta = \begin{cases} 1 & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0 \\ -1 & \text{if } \theta < 0 \end{cases}$$

Stable density is not only support for all of $(-\infty, +\infty)$, but also for a half line. For $0 < \alpha < 1$ and $\beta = 1$ or $\beta = -1$, the stable density is only for a half line.

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